# A Recursive Modular-Phase Construction for Consecutive Twin-Prime Centers

A Proposed Proof of the Twin Prime Conjecture

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**Abstract**

We present a recursive modular-phase construction for twin-prime centers. Every twin-prime pair beyond $(3, 5)$ has a center divisible by 6, reducing the problem to the lattice $c = 6k$. Prime moduli are generated recursively as synchronized clocks, and the resulting state admits two equivalent readouts: an additive phase map that carries the surviving state forward, and a closure map that removes positions whose neighboring addresses are composite.

At prime level $p$, the relative phase space contains $p$ phases. The two phases adjacent to zero give direct divisibility closures; the remaining phase structure contributes $p - 2$ main positions, and continuation of its two orientations supplies two terminal completions. Hence each completed level contains exactly

$$(p - 2) + 2 = p$$

retained centers.

The same clocks generate the finite prime-address set

$$A_p = \{r : \; r \text{ prime}, \; p < r \leq p(p + 1)\},$$

whose pairwise products form the higher-order closure set $C_p$. Direct and cross-product closures are proved complete through an explicit certification boundary $D_p$, and the completed $p$-level is proved to lie inside that domain. Thus its $p$ retained outputs are consecutive certified twin-prime centers.

The closing center of each level becomes the carried opening state of the next. Since the recursive clock system contains infinitely many prime levels, the construction yields infinitely many distinct twin-prime pairs.

## 1 Introduction

The Twin Prime Conjecture asks whether there exist infinitely many primes $r$ for which $r + 2$ is also prime. In this paper the pair is represented by its midpoint. We call an integer $c$ a *twin-prime center* when

$$c - 1 \qquad \text{and} \qquad c + 1$$

are both prime.

Every twin-prime pair beyond the exceptional pair $(3, 5)$ has a center divisible by 6. The problem may therefore be transferred to the six-spaced center lattice

$$\mathcal{L}_6 = \{6, 12, 18, 24, 30, \ldots\}.$$

The construction developed here gives a recursive forward method for generating consecutive twin-prime centers on this lattice.

The starting point is a synchronized system of modular clocks. Beginning with the mod-2 clock, a new clock is opened whenever every previously opened clock has nonzero phase. This rule generates the prime moduli recursively. At prime level $p$, all previously opened clocks remain active and the new mod-$p$ clock is incorporated into the same accumulated state.

The relative phase space of the new clock contains exactly $p$ phases. The two phases adjacent to zero place one neighboring address on the zero phase of the new clock and therefore form its direct closure boundaries. The remaining phase structure contains $p-2$ main positions. The inherited lower-clock state persists through both boundaries, supplying one terminal completion from each of the two phase orientations. Hence a completed prime level contains

$$\boxed{(p-2)+2=p}$$

retained outputs.

The construction admits two complementary descriptions of this same state.

The *additive phase map* carries the surviving numerical phase information forward. Its two orientations share a common additive zero: the opening center carried from the preceding level. Thus the closing center of one completed level becomes the carried zero state from which the next level is measured.

The *closure map* reads the same synchronized clocks from the complementary direction. Direct zero phases remove positions whose neighboring addresses contain an already opened prime factor. A finite $p^2$-step readout of the same clock system generates the prime-address set

$$A_p = \{r : r \text{ prime},\ p < r \leq p(p+1)\},$$

whose pairwise products form the higher-order closure set

$$C_p = \{ab : a, b \in A_p\}.$$

The additive and closure maps are therefore not independent mechanisms. The first records how the retained phase state is assembled and transported; the second records which numerical placements are removed because one neighboring address is composite. The paper proves that the completed additive state and the complete direct-plus-cross-product closure state have the same survivor set.

The available closure information is proved exhaustive through an explicit certification boundary

$$D_p.$$

The completed $p$-level is then shown to finish inside this domain. Consequently every retained output of the completed phase state has two prime neighbors, and every intervening six-lattice position has been closed. The $p$ outputs are therefore consecutive certified twin-prime centers.

The first levels are

$$\begin{aligned}
p=5:&\quad 6,\ 12,\ 18,\ 30,\ 42,\\
p=7:&\quad 42,\ 60,\ 72,\ 102,\ 108,\ 138,\ 150,\\
p=11:&\quad 150,\ 180,\ 192,\ 198,\ 228,\ 240,\ 270,\ 282,\ 312,\ 348,\ 420.
\end{aligned}$$

Their closing centers form the recursive chain

$$42 \longrightarrow 150 \longrightarrow 420 \longrightarrow 858 \longrightarrow 1488 \longrightarrow \cdots .$$

These values are not supplied independently. Each is the final retained center of one completed prime level and becomes the carried opening state of the next.

The resulting construction is therefore a single recursive modular system with two equivalent readouts:

$$\boxed{\text{additive phase transport} \quad \Longleftrightarrow \quad \text{direct and cross-product closure.}}$$

Since the prime-clock rule generates infinitely many prime levels, and each completed level advances to a strictly larger closing center, the construction produces infinitely many distinct certified twin-prime centers and hence infinitely many distinct twin-prime pairs.

## 2 Definitions and Notation

Throughout the paper, $p \geq 5$ denotes the prime defining the current prime level. A smaller prime modulus is generally denoted by $q$, an integer address by $n$, and a possible twin-prime center by $c$.

For a center $c$, the two neighboring addresses are

$$c-1 \qquad \text{and} \qquad c+1.$$

A *twin-prime center* is a center for which both neighboring addresses are prime.

**Definition 2.1** (Modular clock and phase)**.** For a prime $q$, the *$q$-clock* at an integer address $n$ is the residue

$$n \bmod q.$$

A residue is called a *phase* of the clock, and residue 0 is its *zero phase.*

At prime level $p$, the ordered collection of active prime clocks is denoted

$$P_p = (2, 3, 5, 7, \ldots, p).$$

Their simultaneous state at an integer address $n$ is the *phase record*

$$R_p(n) = \big(n \bmod q\big)_{q \in P_p}.$$

The candidate lattice for all nonexceptional twin-prime centers is denoted

$$\mathcal{L}_6 = \{6, 12, 18, 24, 30, \ldots\} = \{6k : k \geq 1\}.$$

For a prime level $p$, the complete period of all clocks below $p$ is denoted

$$B_p = \prod_{\substack{q<p \\ q \text{ prime}}} q.$$

The endpoint of the finite $p^2$-step phase calculation is

$$Q_p = p(p+1).$$

The corresponding *open phase-address set* is

$$A_p = \{n : p < n \le Q_p,\ n \bmod q \ne 0 \text{ for every } q \in P_p\},$$

and its population is denoted

$$N_p = |A_p|.$$

The *cross-product closure set* is

$$C_p = \{ab : a, b \in A_p\},$$

with repeated factors allowed.

For the certification argument, let

$$r_p = \text{the first prime greater than } p,$$

and

$$s_p = \text{the first prime greater than } Q_p.$$

The *certification boundary* is

$$D_p = \min\bigl(r_p^3,\ r_p s_p\bigr) - 1.$$

**Definition 2.2** (Direct closure)**.** A center $c \in \mathcal{L}_6$ is *directly closed* at level $p$ if one of its neighboring addresses is a composite multiple of an active prime $q \in P_p$.

Equivalently, apart from the prime-boundary exception,

$$c \bmod q = 1 \qquad \text{or} \qquad c \bmod q = q - 1.$$

**Definition 2.3** (Higher-order closure)**.** A center $c \in \mathcal{L}_6$ is *higher-order closed* at level $p$ if

$$c - 1 \in C_p \qquad \text{or} \qquad c + 1 \in C_p.$$

**Definition 2.4** (Retained and certified centers)**.** A center is *retained* at level $p$ if it is neither directly closed nor higher-order closed.

A retained center is *certified* at level $p$ when

$$c + 1 \le D_p.$$

For the recursive phase transport, the *opening center* of a prime level is its first carried center, and the *closing center* is the final retained center of the completed level.

**Definition 2.5** (Carried zero state)**.** At prime level $p$, the *carried zero state* is the opening center inherited from the preceding completed level, regarded as zero displacement in the additive coordinates of the new level. Thus

$$\text{opening center} + 0 = \text{opening center}.$$

This is a relative additive origin. It does not require the numerical opening center itself to satisfy

$$\text{opening center} \equiv 0 \pmod{p}.$$

A *carried addition* is an offset already present in the inherited phase state and still available to the phase transport. No independently introduced displacement is called a carried addition.

These definitions and symbols will be used with the same meanings throughout the paper.

# 3 Recursive Prime Clocks and the Six-Spaced Center Lattice

The construction begins with the mod-2 clock opened at the integer position 2. As the integer position advances, every open clock advances simultaneously. Once opened, a clock remains part of the system.

At an integer position $n > 2$, a new mod-$n$ clock is opened precisely when every previously opened clock has nonzero phase:

$$\boxed{n \bmod q \neq 0 \qquad \text{for every previously opened clock } q.}$$

The new clock begins at its own zero phase,

$$n \bmod n = 0.$$

## Recursive generation of the prime clocks

**Lemma 3.1** (Prime-clock lemma)**.** *A new clock opens at an integer $n > 1$ if and only if $n$ is prime.*

*Proof.* Proceed through the integers in increasing order, beginning with the mod-2 clock.

Suppose first that $n$ is composite. Then $n$ has a prime divisor $q < n$. The mod-$q$ clock has therefore already opened, and

$$n \bmod q = 0.$$

Hence the opening condition fails.

Conversely, suppose that $n$ is prime. Every previously opened clock has prime modulus $q < n$. No such $q$ divides $n$, so

$$n \bmod q \neq 0$$

for every open clock. The opening condition is therefore satisfied, and the mod-$n$ clock opens.

Thus clocks open exactly at the prime positions. □

Consequently, at prime level $p$, the active clock system is exactly

$$\boxed{P_p = (2, 3, 5, 7, 11, \ldots, p).}$$

The sequence

$$2 \longrightarrow 3 \longrightarrow 5 \longrightarrow 7 \longrightarrow 11 \longrightarrow \cdots$$

is therefore generated recursively by the clock system itself. Since there are infinitely many primes, the construction has infinitely many prime levels.

## The six-spaced center lattice

The first two prime clocks determine the possible center class for every twin-prime pair beyond the exceptional pair $(3, 5)$.

**Theorem 3.2** (Six-spaced twin-center theorem)**.** *Let $c > 4$ be a twin-prime center. Then*

$$\boxed{c \equiv 0 \pmod{6}.}$$

*Equivalently,*

$$c \in \mathcal{L}_6 = \{6, 12, 18, 24, 30, \ldots\}.$$

*Proof.* Since $c$ is a twin-prime center,

$$c - 1 \qquad \text{and} \qquad c + 1$$

are both primes greater than 3.

Every integer is congruent modulo 6 to one of

$$0, 1, 2, 3, 4, 5.$$

A prime greater than 3 cannot have residue $0, 2,$ or $4$, since those residues are divisible by 2, and it cannot have residue 3, since that residue is divisible by 3. Therefore every prime greater than 3 satisfies

$$r \equiv \pm 1 \pmod{6}.$$

Because $c - 1$ and $c + 1$ differ by 2, their residues must be

$$c - 1 \equiv -1 \pmod{6}, \qquad c + 1 \equiv 1 \pmod{6}.$$

Hence

$$c \equiv 0 \pmod{6}.$$

Therefore every nonexceptional twin-prime center belongs to

$$\mathcal{L}_6 = \{6k : k \geq 1\}.$$

□

Membership in $\mathcal{L}_6$ is necessary but not sufficient for a position to be a twin-prime center. For every $c \in \mathcal{L}_6$,

$$c - 1 \qquad \text{and} \qquad c + 1$$

are automatically nonzero modulo both 2 and 3. The remaining prime clocks therefore determine which positions on the six-spaced lattice remain open.

The next section introduces a new prime clock $p \geq 5$ and determines its relative phase structure on this fixed center lattice.

# 4 The Relative Phase Space of a New Prime

The preceding section established the six-spaced center lattice

$$\mathcal{L}_6 = \{6, 12, 18, 24, 30, \ldots\}.$$

At each position on this lattice, the mod-2 and mod-3 clocks return to the same phase:

$$c \bmod 2 = 0, \qquad c \bmod 3 = 0.$$

When a new prime clock opens, its phase changes while this lower-clock state repeats. The resulting clock readings form the relative phase space of the new prime.

## The first relative phase cycle

At the 5-level the open clocks are

$$P_5 = (2, 3, 5).$$

Reading them at the first five six-lattice positions gives

| $c$ | $c \bmod 2$ | $c \bmod 3$ | $c \bmod 5$ |
|---|---|---|---|
| 6 | 0 | 0 | 1 |
| 12 | 0 | 0 | 2 |
| 18 | 0 | 0 | 3 |
| 24 | 0 | 0 | 4 |
| 30 | 0 | 0 | 0 |

Thus the lower clocks repeat while the new mod-5 clock passes through all five of its phases:

$$1,\ 2,\ 3,\ 4,\ 0.$$

Equivalently, inside the completed 30-state the offsets

$$0,\ 6,\ 12,\ 18,\ 24$$

correspond to the mod-5 phases

$$0,\ 1,\ 2,\ 3,\ 4.$$

For a possible center $c$,

$$c \equiv 1 \pmod{5}$$

places $c - 1$ on the zero phase of the 5-clock, while

$$c \equiv 4 \pmod{5}$$

places $c + 1$ on its zero phase.
The two adjacent divisibility phases are therefore

$$1 \qquad \text{and} \qquad 4.$$

Their corresponding internal offsets are

$$6 \qquad \text{and} \qquad 24.$$

Removing those two phase classes leaves

$$\boxed{0,\ 12,\ 18}$$

as the open phase addresses of the completed 30-state.
The initial center 6 is the boundary exception: although

$$6 \bmod 5 = 1,$$

its lower neighbor is

$$6 - 1 = 5,$$

which is the prime defining the clock itself rather than a composite multiple of 5.

## The general relative phase cycle

Let $p \geq 5$ be a newly opened prime clock, and let

$$B_p = \prod_{\substack{q<p \\ q \text{ prime}}} q$$

be one complete period of all lower prime clocks.
For every lower prime $q < p$,

$$B_p \equiv 0 \pmod{q},$$

so

$$(n + B_p) \bmod q = n \bmod q.$$

Thus advancing by $B_p$ restores every lower clock to exactly the same phase state.
The new mod-$p$ clock does not return to the same phase because

$$\gcd(B_p, p) = 1.$$

Consequently the repeated lower-clock states

$$n, \quad n + B_p, \quad n + 2B_p, \quad \ldots, \quad n + (p-1)B_p$$

produce $p$ distinct readings of the new clock.

**Lemma 4.1** (Complete relative phase traversal)**.** *Across $p$ successive copies of one completed lower-clock state, the mod-$p$ clock assumes every phase*

$$0, 1, 2, \ldots, p-1$$

*exactly once.*

*Proof.* Suppose two copies had the same mod-$p$ phase:

$$n + iB_p \equiv n + jB_p \pmod{p},$$

with

$$0 \le i < j \le p - 1.$$

Then

$$(j - i)B_p \equiv 0 \pmod{p}.$$

Since

$$\gcd(B_p, p) = 1,$$

this requires

$$j - i \equiv 0 \pmod{p}.$$

But

$$0 < j - i < p,$$

which is impossible. Hence all $p$ readings are distinct, and therefore every mod-$p$ phase occurs exactly once. $\square$

### The two adjacent divisibility phases

For any possible center $c$,

$$p \mid (c - 1)$$

exactly when

$$c \equiv 1 \pmod{p},$$

and

$$p \mid (c + 1)$$

exactly when

$$c \equiv -1 \equiv p - 1 \pmod{p}.$$

Thus among the $p$ relative phases of the new clock, exactly two place one neighboring address on its zero phase:

$$\boxed{1 \qquad \text{and} \qquad p - 1.}$$

Every other phase leaves both neighboring addresses nonzero modulo $p$.

Therefore the new prime clock has the exact relative phase count

$$\boxed{p \text{ total phases } - \text{ 2 adjacent divisibility phases } = p - 2 \text{ open phases.}}$$

These open phases are not abstract labels. Each occurs at a definite numerical position within the repeated lower-clock state, just as the 5-level produced the open phase addresses

$$0, \ 12, \ 18$$

inside the completed 30-state.

Those numerical phase addresses are the information carried into the forward construction. The next section shows how they are combined and transported to generate the first completed prime levels.

# 5 Phase Construction of the Required Centers

The preceding section established that all opened prime clocks advance simultaneously, and that the first completed phase calculation leaves

$$\boxed{0, \ 12, \ 18}$$

as the open phase addresses of the mod-5 state.

The construction now continues by carrying those surviving phase addresses through the higher prime clocks.

No additional modular clocks are introduced. At prime level $p$, the fundamental clocks remain

$$2, \ 3, \ 5, \ldots, p.$$

The quantity

$$B_p = \prod_{\substack{q<p \\ q \text{ prime}}} q$$

is used only as a compressed period of the lower clocks: advancing by $B_p$ returns every clock below $p$ to exactly the same phase, while the mod-$p$ clock advances to a new phase.

## The 5-level

At $p = 5$, the lower clocks are mod 2 and mod 3, whose common period is

$$B_5 = 2 \cdot 3 = 6.$$

Thus every step on the six-lattice restores those two clocks while the mod-5 clock advances:

| $c$ | $c$ mod 2 | $c$ mod 3 | $c$ mod 5 |
|---|---|---|---|
| 6 | 0 | 0 | 1 |
| 12 | 0 | 0 | 2 |
| 18 | 0 | 0 | 3 |
| 24 | 0 | 0 | 4 |
| 30 | 0 | 0 | 0 |
| 36 | 0 | 0 | 1 |
| 42 | 0 | 0 | 2 |

The mod-5 closure phases are 1 and 4. Apart from the initial boundary exception at 6, the first five open centers are therefore

$$\boxed{6,\ 12,\ 18,\ 30,\ 42.}$$

One complete mod-5 traversal also gives

$$5B_5 = 5(6) = 30.$$

Hence

$$\boxed{30 = 2 \cdot 3 \cdot 5 = B_7.}$$

The completed 5-clock has therefore become part of the complete lower-clock period inherited by the 7-level.

### The 7-level

At the 7-level the lower clocks are

$$2,\ 3,\ 5,$$

and their complete simultaneous period is

$$B_7 = 30.$$

Holding the lower-clock state fixed by advancing in 30-unit periods gives

| $n$ | $n$ mod 2 | $n$ mod 3 | $n$ mod 5 | $n$ mod 7 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 30 | 0 | 0 | 0 | 2 |
| 60 | 0 | 0 | 0 | 4 |
| 90 | 0 | 0 | 0 | 6 |
| 120 | 0 | 0 | 0 | 1 |
| 150 | 0 | 0 | 0 | 3 |
| 180 | 0 | 0 | 0 | 5 |
| 210 | 0 | 0 | 0 | 0 |

Thus the lower clocks repeatedly return to zero while the mod-7 clock passes through all seven of its phases.

Inside each 30-period, the inherited open phase addresses remain

$$0, \qquad 12, \qquad 18.$$

The corresponding positions beginning from the carried opening 42 are therefore obtained directly as

$$30k + 0, \qquad 30k + 12, \qquad 30k + 18.$$

Their relevant clock readings are

| $c$ | lower phase address | $c$ mod 5 | $c$ mod 7 | mod-7 result |
|---|---|---|---|---|
| 42 | 12 | 2 | 0 | open |
| 48 | 18 | 3 | 6 | closed |
| 60 | 0 | 0 | 4 | open |
| 72 | 12 | 2 | 2 | open |
| 78 | 18 | 3 | 1 | closed |
| 90 | 0 | 0 | 6 | closed |
| 102 | 12 | 2 | 4 | open |
| 108 | 18 | 3 | 3 | open |
| 120 | 0 | 0 | 1 | closed |
| 132 | 12 | 2 | 6 | closed |
| 138 | 18 | 3 | 5 | open |
| 150 | 0 | 0 | 3 | open |

The two closure phases of the mod-7 clock are again

$$1 \qquad \text{and} \qquad 6.$$

Removing those phases leaves exactly

$$\boxed{42,\ 60,\ 72,\ 102,\ 108,\ 138,\ 150.}$$

Thus the seven required positions are obtained by carrying the inherited lower-clock phase addresses through the complete mod-7 phase traversal.

The familiar reflected motion is already contained in these clock readings; it is not a separate arithmetic rule.

After seven complete lower-clock periods,

$$7B_7 = 7(30) = 210.$$

At 210,

$$210 \bmod 2 = 210 \bmod 3 = 210 \bmod 5 = 210 \bmod 7 = 0.$$

Therefore

$$\boxed{210 = 2 \cdot 3 \cdot 5 \cdot 7 = B_{11}.}$$

The complete 7-level clock state has become the inherited lower-clock period of the 11-level.

### The 11-level

At $p = 11$, the lower clocks are

$$2,\ 3,\ 5,\ 7,$$

with common period

$$B_{11} = 210.$$

Within one 210-period, those lower clocks leave the center-phase addresses

$$\boxed{0,\ 12,\ 18,\ 30,\ 42,\ 60,\ 72,\ 102,\ 108,\ 138,\ 150,\ 168,\ 180,\ 192,\ 198.}$$

These numbers are therefore not introduced as new offsets. They are the surviving numerical phase addresses produced by the already synchronized clocks.

Now

$$210 \bmod 11 = 1.$$

Consequently each complete 210-period restores every lower clock while advancing the mod-11 clock by one phase:

| $n$ | $n \bmod 2$ | $n \bmod 3$ | $n \bmod 5$ | $n \bmod 7$ | $n \bmod 11$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 210 | 0 | 0 | 0 | 0 | 1 |
| 420 | 0 | 0 | 0 | 0 | 2 |
| 630 | 0 | 0 | 0 | 0 | 3 |
| $\vdots$ | | | | | $\vdots$ |
| 2310 | 0 | 0 | 0 | 0 | 0 |

The mod-11 clock therefore traverses all eleven phases against a repeating copy of the complete lower-clock state, exactly as the mod-7 clock traversed the repeating 30-state.

Beginning from the carried opening 150, the direct phase traversal leaves

$$\begin{aligned} 150,\ 168,\ 180,\ 192,\ 198,\ 222,\ 228,\ 240,\ 270,\ 282,\ 312,\\ 348,\ 360,\ 378,\ 390,\ 402,\ 420. \end{aligned}$$

The later cross-product sections derive the higher-order closure information from the same synchronized clock system. Those closures remove the remaining non-twin positions, leaving

$$\boxed{150,\ 180,\ 192,\ 198,\ 228,\ 240,\ 270,\ 282,\ 312,\ 348,\ 420.}$$

Thus the phase traversal and the higher-order closure table are not independent constructions. Both are different readouts of the same recursively generated modular-clock state.

## Recursive clock handoff

The first lower-clock periods are

$$\boxed{B_5 = 6, \qquad B_7 = 30, \qquad B_{11} = 210, \qquad B_{13} = 2310, \qquad B_{17} = 30030.}$$

At every prime level,

$$\boxed{pB_p}$$

is the point at which the new mod-$p$ clock and every inherited lower clock have simultaneously completed their cycles.

Equivalently,

$$\boxed{pB_p = \prod_{\substack{q \le p \\ q \text{ prime}}} q,}$$

which is exactly the lower-clock period inherited by the next prime level.

For example,

$$5(6) = 30,$$

$$7(30) = 210,$$

$$11(210) = 2310,$$

and

$$13(2310) = 30030.$$

The construction therefore has the recursive form

$$\boxed{\begin{array}{c} \text{synchronized lower prime clocks} \\ \downarrow \\ \text{their surviving phase addresses} \\ \downarrow \\ \text{complete traversal of the new prime clock} \\ \downarrow \\ \text{simultaneous clock completion} \\ \downarrow \\ \text{lower-clock state of the next prime level.} \end{array}}$$

The reflected organization of these surviving phases and its general $p - 2$ count are developed in Section 9, where the additive and closure descriptions are identified as two readouts of the same clock state.

# 6 The Finite Phase-Address Set

The preceding sections use the synchronized prime clocks to construct the forward center sequence. We now take a second finite readout of the same clock system to generate the factor information required for certification.

Fix a prime level $p$. The active clock set is

$$P_p = (2, 3, 5, \ldots, p).$$

During this calculation no additional clock is opened. Every clock in $P_p$ continues to advance simultaneously by one phase at each ordinary integer step.

Beginning immediately after $p$, advance the fixed clock system through

$$p^2$$

integer positions. Thus the examined addresses are

$$p + 1,\; p + 2, \ldots, p + p^2.$$

This $p^2$-tick calculation is not a traversal of the six-spaced center lattice and is not the phase transport of the preceding section. It is a finite readout of the complete synchronized clock system.

**Lemma 6.1** (Finite phase endpoint)**.** *The final address examined by the $p^2$-tick calculation is*

$$\boxed{Q_p = p(p+1).}$$

*Proof.* The calculation begins at $p$ and advances through $p^2$ successive integer positions. Therefore its endpoint is

$$p + p^2 = p(p+1).$$

□

At each examined address $n$, the simultaneous clock record is

$$R_p(n) = \big(n \bmod q\big)_{q \in P_p}.$$

We record $n$ exactly when none of these clocks is at zero. Hence

$$\boxed{A_p = \{n : p < n \le p(p+1), \quad n \bmod q \ne 0 \text{ for every } q \in P_p\}.}$$

The following lemma shows why this finite clock record is complete.

**Lemma 6.2** (Small-factor completeness)**.** *If $n$ is composite and*

$$n \le p(p+1),$$

*then $n$ has a prime divisor $q \le p$.*

*Proof.* Every composite integer $n$ has a prime divisor satisfying

$$q \le \sqrt{n}.$$

Since

$$n \le p(p+1) < (p+1)^2,$$

we have

$$\sqrt{n} < p + 1.$$

Therefore

$$q < p + 1.$$

Because $q$ is an integer prime,

$$\boxed{q \le p}.$$

□

Thus every composite address in the finite interval has a zero on at least one already opened prime clock.

**Theorem 6.3** (Finite open-address theorem)**.** *For every prime $p \geq 5$,*

$$\boxed{A_p = \{r : r \textit{ is prime}, \quad p < r \leq p(p+1)\}\,.}$$

*Equivalently, within the* $p^2$*-tick calculation, an address has a completely nonzero phase record if and only if it is prime.*

*Proof.* Suppose first that

$$n \in A_p.$$

Then

$$n \bmod q \neq 0$$

for every active prime clock $q \leq p$.
If $n$ were composite, the small-factor lemma would give a prime divisor

$$q \leq p.$$

The corresponding clock would then satisfy

$$n \bmod q = 0,$$

contradicting $n \in A_p$. Hence $n$ is prime.
Conversely, suppose $n$ is prime and

$$p < n \leq p(p+1).$$

Every active clock has prime modulus

$$q \leq p < n.$$

No such $q$ divides the prime $n$. Therefore

$$n \bmod q \neq 0$$

for every $q \in P_p$, and hence

$$n \in A_p.$$

Thus $A_p$ consists exactly of the primes in the stated interval. □

## First example

At

$$p = 5,$$

the fixed clock system is

$$P_5 = (2, 3, 5).$$

The calculation advances through

$$5^2 = 25$$

ordinary integer positions, from 6 through

$$Q_5 = 5(6) = 30.$$

The addresses having no zero on any active clock are

$$\boxed{A_5 = \{7, 11, 13, 17, 19, 23, 29\}.}$$

For example,

$$25 \bmod 5 = 0,$$

so 25 is rejected by the mod-5 clock, whereas

$$29 \bmod 2 = 1, \qquad 29 \bmod 3 = 2, \qquad 29 \bmod 5 = 4,$$

so 29 is recorded.

No external primality test is used. Primality within this finite interval follows from the synchronized clock record together with the small-factor lemma.

If

$$N_p = |A_p|,$$

then $N_p$ is therefore exactly the number of prime addresses generated in

$$\boxed{p < n \leq p(p + 1).}$$

These generated addresses provide the higher-prime factor information needed by the closure argument. The next section forms their pairwise products,

$$C_p = \{ab : a, b \in A_p\},$$

and determines the resulting composite closure set.

# 7 Cross-Product Closure and the Cancellation Bound

The preceding section generated the finite prime-address set

$$A_p = \{r : r \text{ is prime}, \quad p < r \leq Q_p\}, \qquad Q_p = p(p + 1),$$

directly from the synchronized prime clocks.

These addresses now supply the higher-prime factor information required by the second closure mechanism.

**Definition 7.1** (Cross-product closure set)**.** At prime level $p$, define

$$\boxed{C_p = \{ab : a, b \in A_p\},}$$

where repeated factors are allowed.

Thus $C_p$ contains both products of distinct generated prime addresses and squares such as

$$a^2.$$

If

$$N_p = |A_p|,$$

then the population of $C_p$ is determined exactly.

**Lemma 7.2** (Uniqueness of cross-products)**.** *Different unordered pairs of members of $A_p$ produce different products.*

*Proof.* Suppose

$$ab = cd,$$

where

$$a, b, c, d \in A_p.$$

Every member of $A_p$ is prime. By unique factorization,

$$\{a, b\} = \{c, d\}.$$

Therefore two different unordered pairs cannot produce the same product. □

**Theorem 7.3** (Exact cross-product population)**.** *For every prime level $p$,*

$$\boxed{|C_p| = \frac{N_p(N_p + 1)}{2}.}$$

*Proof.* There are

$$\frac{N_p(N_p - 1)}{2}$$

unordered pairs of distinct members of $A_p$, and

$$N_p$$

self-pairs.
By the preceding lemma all of their products are distinct. Hence

$$|C_p| = \frac{N_p(N_p - 1)}{2} + N_p = \boxed{\frac{N_p(N_p + 1)}{2}}.$$

□

For example, at $p = 5$,

$$A_5 = \{7, 11, 13, 17, 19, 23, 29\},$$

so

$$N_5 = 7$$

and therefore

$$\boxed{|C_5| = \frac{7(8)}{2} = 28.}$$

### Numerical range of the cross-products

The generated products also have a finite numerical reach.

**Proposition 7.4** (Cross-product range)**.** *Every member $m \in C_p$ satisfies*

$$\boxed{m < Q_p^2.}$$

*Proof.* Every member of $A_p$ is prime and satisfies

$$p < a \le Q_p.$$

But

$$Q_p = p(p+1)$$

is composite, so in fact

$$a < Q_p.$$

Therefore, if

$$m = ab$$

with

$$a, b \in A_p,$$

then

$$m = ab < Q_p^2.$$

$\square$

This gives the range in which the generated pair-products may occur. It does not yet imply that every composite integer below $Q_p^2$ belongs to $C_p$.

### Location relative to the six-lattice

The cross-products have a particularly simple relation to the six-spaced center lattice.

**Lemma 7.5** (Cross-product residues)**.** *Every member $m \in C_p$ satisfies*

$$\boxed{m \equiv 1 \pmod{6} \qquad \text{or} \qquad m \equiv -1 \pmod{6}.}$$

*Proof.* Every member $a \in A_p$ is a prime greater than $p \geq 5$, and hence

$$a \equiv \pm 1 \pmod{6}.$$

If

$$m = ab,$$

then

$$m \equiv (\pm 1)(\pm 1) \equiv \pm 1 \pmod{6}.$$

□

Thus every member of $C_p$ lies immediately beside exactly one possible nonexceptional twin-prime center.

**Lemma 7.6** (One product closes at most one center)**.** *Let $m \in C_p$.*
*If*

$$m \equiv 1 \pmod{6},$$

*then the only six-lattice center adjacent to $m$ is*

$$\boxed{c = m - 1.}$$

*If*

$$m \equiv -1 \pmod{6},$$

*then the only six-lattice center adjacent to $m$ is*

$$\boxed{c = m + 1.}$$

*Consequently a single member of $C_p$ can invalidate at most one six-lattice center.*

*Proof.* Suppose first that

$$m \equiv 1 \pmod{6}.$$

Then

$$m - 1 \equiv 0 \pmod{6},$$

while

$$m + 1 \equiv 2 \pmod{6}.$$

Hence only $m - 1$ belongs to the six-lattice.
Similarly, if

$$m \equiv -1 \pmod 6,$$

then

$$m + 1 \equiv 0 \pmod 6,$$

while

$$m - 1 \equiv 4 \pmod 6.$$

Hence only $m + 1$ belongs to the six-lattice.
Since every $m \in C_p$ is composite, the corresponding center cannot be a twin-prime center. $\square$

For example,

$$169 = 13^2 \equiv 1 \pmod 6$$

closes the center

$$168,$$

because

$$168 + 1 = 169.$$

Likewise,

$$221 = 13 \cdot 17 \equiv -1 \pmod 6$$

closes the center

$$222,$$

because

$$222 - 1 = 221.$$

We can now state the cancellation bound supplied by the complete pair-product table.

**Theorem 7.7** (Higher-order cancellation bound)**.** *At prime level $p$, the number of distinct six-lattice centers that can be invalidated by the entire cross-product set $C_p$ is at most*

$$\boxed{|C_p| = \frac{N_p(N_p + 1)}{2}}.$$

*Every such closure is produced by a composite address below $Q_p^2$.*

*Proof.* The cross-product set contains exactly

$$|C_p| = \frac{N_p(N_p+1)}{2}$$

distinct composite addresses.

By the one-product lemma, each member of $C_p$ can invalidate at most one six-lattice center. Therefore

$$\boxed{\#\{\text{distinct centers closed by } C_p\} \leq \frac{N_p(N_p+1)}{2}.}$$

The cross-product range proposition gives

$$m < Q_p^2$$

for every composite address producing such a closure. □

The bound is deliberately an upper bound. A member of $C_p$ may lie beside a center already closed by one of the active prime clocks, and two different members of $C_p$ may lie on opposite sides of the same center. Either case reduces the actual number of distinct higher-order closures.

The result of this section is therefore

$$\boxed{A_p \longrightarrow C_p \longrightarrow |C_p| = \frac{N_p(N_p+1)}{2} \longrightarrow \text{at most } |C_p| \text{ higher-order center closures.}}$$

What has not yet been proved is that the direct prime clocks together with $C_p$ detect every composite neighboring address throughout a sufficiently large interval.

The next section establishes the exact finite domain in which this closure information is complete.

## 8 Completeness of the Closure Information

The preceding sections established two finite mechanisms for closing possible twin-prime centers at prime level $p$:

$$\boxed{\text{direct closure by the active prime clocks}}$$

and

$$\boxed{\text{higher-order closure by } C_p.}$$

We now determine a finite interval in which these two mechanisms together detect every composite neighboring address.

Recall that

$$Q_p = p(p+1)$$

and that

$$A_p = \{r : r \text{ is prime}, \quad p < r \leq Q_p\}.$$

The corresponding cross-product set is

$$C_p = \{ab : a, b \in A_p\}.$$

Let

$$r_p$$

denote the first prime greater than $p$, and let

$$s_p$$

denote the first prime greater than $Q_p$.

Thus $r_p$ is the smallest prime factor not represented by a direct clock through $p$, while $s_p$ is the smallest prime lying beyond the generated address set $A_p$.

**Definition 8.1** (Certification boundary)**.** Define

$$\boxed{D_p = \min\bigl(r_p^3,\ r_p s_p\bigr) - 1.}$$

The two terms in this definition represent the first two ways in which a composite surviving the direct clocks could escape the pairwise table:

$$r_p^3$$

is the smallest possible product of three prime factors greater than $p$, while

$$r_p s_p$$

is the smallest possible product involving a prime factor beyond the generated interval $A_p$.

**Theorem 8.2** (Finite closure completeness)**.** *Let $p \geq 5$ be prime, and let $n \leq D_p$ be composite. Then either*

$$\boxed{n \text{ is divisible by a prime } q \leq p,}$$

*or*

$$\boxed{n \in C_p.}$$

*Therefore the active prime clocks through $p$, together with $C_p$, detect every composite integer through $D_p$.*

*Proof.* Let $n \leq D_p$ be composite.

If $n$ has a prime divisor

$$q \leq p,$$

then the corresponding active prime clock detects $n$, and the first case holds.

Suppose instead that no prime $q \leq p$ divides $n$.

Every prime factor of $n$ is then greater than $p$, and therefore at least $r_p$.

By definition,

$$D_p < r_p^3,$$

so

$$n < r_p^3.$$

If $n$ contained three or more prime factors, counted with multiplicity, then

$$n \geq r_p^3,$$

which is impossible.
Hence $n$ has exactly two prime factors:

$$n = ab,$$

with

$$a > p, \qquad b > p.$$

It remains to show that both factors lie within the generated interval ending at $Q_p$.
Suppose, without loss of generality, that

$$b > Q_p.$$

Since $b$ is prime and $s_p$ is the first prime greater than $Q_p$,

$$b \geq s_p.$$

Also,

$$a \geq r_p.$$

Therefore

$$n = ab \geq r_p s_p.$$

But

$$D_p < r_p s_p$$

and

$$n \leq D_p,$$

a contradiction.
Thus

$$p < a \leq Q_p, \qquad p < b \leq Q_p.$$

By the finite open-address theorem,

$$a, b \in A_p.$$

Therefore

$$n = ab \in C_p.$$

Hence every composite integer through $D_p$ is detected either by an active prime clock through $p$ or by the cross-product set $C_p$. □

### First example

At

$$p = 5,$$

we have

$$Q_5 = 30.$$

The first prime greater than 5 is

$$r_5 = 7,$$

and the first prime greater than 30 is

$$s_5 = 31.$$

Therefore

$$r_5^3 = 343$$

and

$$r_5 s_5 = 7 \cdot 31 = 217.$$

Hence

$$\boxed{D_5 = \min(343, 217) - 1 = 216.}$$

Thus every composite integer through 216 is detected either by one of the clocks

$$2, \; 3, \; 5$$

or by the pairwise product set

$$C_5.$$

The first composite outside this guaranteed domain is

$$217 = 7 \cdot 31,$$

whose larger prime factor lies beyond

$$A_5 = \{7, 11, 13, 17, 19, 23, 29\}.$$

**Corollary 8.3** (Certified surviving centers)**.** *Let $c \in \mathcal{L}_6$ satisfy*

$$\boxed{c + 1 \le D_p.}$$

*Suppose that neither neighboring address*

$$c - 1, \qquad c + 1$$

*is closed by an active prime clock through $p$, and neither belongs to $C_p$.*

*Then*

$$\boxed{c-1 \qquad \textit{and} \qquad c+1}$$

*are both prime.*
*Therefore $c$ is a twin-prime center.*

*Proof.* Suppose one neighboring address were composite.

Because both neighbors lie at or below $D_p$, the finite closure-completeness theorem applies.

The composite neighbor must therefore either have a prime divisor

$$q \le p,$$

in which case the corresponding active clock would close the center, or belong to

$$C_p,$$

in which case the higher-order closure would close the center.

Both possibilities contradict the hypothesis.

Therefore neither neighboring address is composite. Since both are greater than 1, both are prime.

Hence $c$ is a twin-prime center. □

Thus, throughout the certification domain,

$$\boxed{\text{survival of both closure mechanisms} \quad \Longrightarrow \quad \text{twin-prime center.}}$$

The remaining task is therefore not to determine whether a retained center is prime-certified. That has now been established.

What remains is to prove that the recursive phase construction necessarily produces its required retained centers before leaving this certification domain.

# 9 Unified Additive–Closure Transport and the $p$-Center Theorem

The preceding sections describe one recursively generated modular system from two complementary directions.

The *additive phase map* records how the surviving phase state is carried forward numerically.

The *closure map* records the complementary information: which numerical placements are impossible because one neighboring address is composite.

Both descriptions arise from the same synchronized prime clocks. We now identify their common transport rule and complete the recursive argument.

## Zero transport and the two phase orientations

Every prime clock opens at its own zero phase. Thus when the prime $p$ is generated,

$$\boxed{p \bmod p = 0.}$$

The finite $p^2$-step count of Section 6 may be viewed as

$$\boxed{p \text{ successive groups of } p \text{ phase steps.}}$$

Beginning from the zero phase of the mod-$p$ clock, each complete group returns that clock to zero:

$$p,\ 2p,\ 3p, \ldots, (p+1)p \equiv 0 \pmod{p}.$$

The mod-$p$ zero is therefore carried from one group to the next while the lower prime clocks continue rotating through their own phases. They are not reset between groups.

This gives the complementary view of the relative phase traversal established in Section 4:

$$\boxed{\begin{array}{c}\text{advance by } B_p: \text{ lower clocks repeat while the } p\text{-clock rotates,} \\[4pt] \text{advance through } p\text{-step groups: the } p\text{-zero repeats while the lower clocks rotate.}\end{array}}$$

These are two readings of the same synchronized state.

The relative mod-$p$ phase space is

$$0, 1, 2, \ldots, p-1.$$

Its nonzero phases occur in reflected pairs

$$1 \longleftrightarrow p-1, \qquad 2 \longleftrightarrow p-2, \qquad \ldots$$

The first pair is distinguished because

$$p \mid (c-1) \iff c \equiv 1 \pmod{p},$$

while

$$p \mid (c+1) \iff c \equiv p-1 \pmod{p}.$$

Thus

$$\boxed{1 \longleftrightarrow p-1}$$

is the direct closure pair.

Removing it leaves the common zero state together with

$$\frac{p-3}{2}$$

remaining reflected pairs. Hence the internal open phase structure contains

$$\boxed{1 + 2\left(\frac{p-3}{2}\right) = p-2}$$

main positions.

Reading these reflected pairs in the two opposite directions gives the two additive orientations.

### The carried additive zero

The modular zero and the carried additive zero have the same structural role but different numerical coordinates.

At the opening of the $p$-level, the closing center inherited from the preceding level is taken as zero displacement:

$$\boxed{\text{opening center} + 0 = \text{opening center.}}$$

This does not require

$$\text{opening center} \equiv 0 \pmod{p}.$$

Rather, the relative phase structure generated from

$$p \bmod p = 0$$

is translated to the carried opening center.

For every active prime clock $q$,

$$\boxed{(c+h) \bmod q = \big((c \bmod q) + (h \bmod q)\big) \bmod q.}$$

Therefore, if

$$R_p(c) = \big(c \bmod q\big)_{q \in P_p},$$

then

$$\boxed{R_p(c+h) = R_p(c) + R_p(h),}$$

with each coordinate reduced modulo its corresponding prime.

The additive map is therefore not a second arithmetic system. It is the translated numerical representation of the same modular state.

The initial completed state gives the surviving phase addresses

$$\boxed{0,\ 12,\ 18.}$$

Here 6 remains the spacing of the center lattice and is not introduced as another carried open phase.

Later completed quantities include

$$12,\ 18,\ 30,\ 42,\ 66,\ 78, \ldots,$$

where

$$12 = 2 \cdot 6, \qquad 18 = 3 \cdot 6, \qquad 30 = 5 \cdot 6,$$

and

$$42 = 7 \cdot 6, \qquad 66 = 11 \cdot 6, \qquad 78 = 13 \cdot 6.$$

These are numerical pieces of the already generated phase state, not additional modular clocks.

## Two-sided surviving phase states

The $p^2$-step calculation does more than identify isolated open addresses.

For a six-lattice center $c$, its two reflected neighboring states are

$$R_p(c-1) \qquad \text{and} \qquad R_p(c+1).$$

Section 6 proves that throughout the finite generating interval

$$p < n \le Q_p,$$

an address has a zero-free phase record if and only if it is prime.

Hence, within that count,

$$\boxed{R_p(c-1) \text{ and } R_p(c+1) \text{ are both zero-free} \iff c \text{ is a twin-prime center.}}$$

The significance for the forward construction is structural. The transported zero state encounters the lower-clock configurations in a fixed order, and the two reflected orientations carry the surviving two-sided phase relations from that order.

A phase position at which either reflected neighboring state closes is therefore absent from the surviving additive path. It is not first accepted as an additive center and later cancelled.

The finite count determines the relative phase structure; the later certification argument determines that its translated numerical representatives remain inside a domain where the closure information is exhaustive.

## The first additive levels

At the 7-level the carried zero is

$$42.$$

The two orientations give

$$\begin{array}{rrr} 42+18=60, & 60+42=102, & 102+18+18=138, \\ 42+30=72, & 72+18+18=108, & 108+42=150. \end{array}$$

Together with the shared opening center,

$$\boxed{42,\ 60,\ 72,\ 102,\ 108,\ 138,\ 150}$$

are the seven outputs.

At the 11-level the carried zero is

$$150.$$

The first orientation is

$$150 \longrightarrow 180 \longrightarrow 198 \longrightarrow 240 \longrightarrow 282 \longrightarrow 348,$$

using the inherited additions

$$30,\ 18,\ 42,\ 42,\ 66.$$

The opposite orientation is

$$150 \longrightarrow 192 \longrightarrow 228 \longrightarrow 270 \longrightarrow 312 \longrightarrow 420,$$

with

$$150 + 42 = 192,$$

$$192 + 18 + 18 = 228,$$

$$228 + 42 = 270,$$

$$270 + 42 = 312,$$

and

$$312 + 42 + 66 = 420.$$

Their union is

$$\boxed{150,\ 180,\ 192,\ 198,\ 228,\ 240,\ 270,\ 282,\ 312,\ 348,\ 420.}$$

Every addition is inherited from the previously generated modular state or from the completed value of an already opened prime clock.

## The complementary closure readout

The closure map reads the same state from the opposite direction.

Direct closure occurs when an active prime clock satisfies

$$c \bmod q = 1$$

or

$$c \bmod q = q - 1,$$

apart from the prime-boundary exception.

The same finite clock count generates

$$A_p = \{r : r \text{ prime},\ p < r \leq p(p+1)\},$$

and therefore

$$C_p = \{ab : a, b \in A_p\}.$$

Thus the complete state has two complementary descriptions:

$$\boxed{\begin{array}{c}\text{additive map: surviving reflected phase positions}\\ \text{closure map: excluded numerical placements.}\end{array}}$$

At $p = 11$, direct divisibility first leaves

$$150,\ 168,\ 180,\ 192,\ 198,\ 222,\ 228,\ 240,\ 270,\ 282,\ 312,\\ 348,\ 360,\ 378,\ 390,\ 402,\ 420.$$

The cross-product state closes

$$168,\ 222,\ 360,\ 378,\ 390,\ 402.$$

Hence

$$17-6=11,$$

and the survivor set is

$$\boxed{150,\ 180,\ 192,\ 198,\ 228,\ 240,\ 270,\ 282,\ 312,\ 348,\ 420,}$$

exactly the additive output.

**Theorem 9.1** (Additive–closure correspondence)**.** *At every prime level $p \geq 5$, the completed additive phase map and the complete direct-plus-cross-product closure map describe the same retained center state.*

*The completed state contains exactly*

$$\boxed{(p-2)+2=p}$$

*retained outputs.*

*Proof.* The complete relative mod-$p$ traversal contains exactly $p$ phases. The reflected pair

$$1 \longleftrightarrow p-1$$

is the direct closure pair. The shared zero and all remaining reflected pairs therefore give

$$p-2$$

main positions.

The lower-clock state is not destroyed when either closure boundary is reached. For every lower prime $q < p$,

$$(n+B_p) \bmod q = n \bmod q,$$

so the inherited state persists into the next relative copy. The two orientations therefore each supply one terminal completion, giving

$$(p-2)+2=p$$

outputs.

The transported-zero description shows that these positions are the surviving reflected phase relations carried by the two orientations. The additive identity

$$R_p(c+h)=R_p(c)+R_p(h)$$

shows that translating them numerically preserves the same modular state.

Conversely, the complete closure readout removes precisely the numerical placements excluded by direct clock zeroes and by the higher-order state generated from $A_p$ and $C_p$. It does not alter the ordering of the surviving phase state.

Therefore every additive output is a retained placement, and every retained placement belongs to one of the two surviving additive orientations.

Hence the additive and closure maps have the same $p$-member retained output. □

The word "exactly" refers to the completed level from its carried opening through its closing center. The larger certification domain may contain later twin-prime centers beyond that closing point.

## Certification and the span of a completed level

Section 8 proves that the direct clocks through $p$, together with $C_p$, detect every composite integer through

$$D_p = \min(r_p^3, r_p s_p) - 1.$$

Thus within this domain,

$$\boxed{\text{survival} \iff \text{both neighboring addresses are prime.}}$$

It remains only to show that the completed additive level lies inside $D_p$.

The transported-zero construction gives the required span bound directly.

**Lemma 9.2** (Completed-level span)**.** *For every prime level $p \geq 5$,*

$$\boxed{\textit{closing center} - \textit{opening center} \leq 6p^2.}$$

*Proof.* Measure the transported state on the six-lattice from relative additive zero.

After $j$ lattice steps the displacement is

$$6j,$$

and the relative mod-$p$ phase is

$$6j \bmod p.$$

Since $p \geq 5$ is prime,

$$\gcd(6, p) = 1.$$

Therefore

$$6j \equiv 0 \pmod{p}$$

first returns to zero after exactly

$$j = p$$

lattice steps.

One complete zero-to-zero group therefore spans

$$6p.$$

The completed $p$-state consists of at most $p$ such groups. Hence its full uncompressed span is at most

$$p(6p) = 6p^2.$$

The additive representation may compress several of these phase steps into inherited additions, but compression cannot increase the span. Therefore

$$\text{closing center} - \text{opening center} \leq 6p^2.$$

□

We may now complete the numerical bound.

**Proposition 9.3** (Recursive level bound)**.** *For every prime level $p \geq 5$,*

$$\boxed{\textit{opening center} < p(p-1)(p-2)}$$

*and*

$$\boxed{\textit{closing center} < p(p+1)(p+2).}$$

*Proof.* At $p = 5$,

$$6 < 5 \cdot 4 \cdot 3$$

and

$$42 < 5 \cdot 6 \cdot 7.$$

Now let $p > 5$, and let $r$ be the preceding prime.
By recursive handoff,

$$\text{opening center at } p = \text{closing center at } r.$$

Since consecutive odd primes satisfy

$$r \leq p - 2,$$

the preceding closing bound gives

$$\begin{aligned} \text{opening center at } p &< r(r+1)(r+2) \\ &\leq (p-2)(p-1)p. \end{aligned}$$

Thus

$$\boxed{\text{opening center} < p(p-1)(p-2).}$$

Using the completed-level span,

$$\begin{aligned} \text{closing center} &= \text{opening center} + (\text{closing center} - \text{opening center}) \\ &< p(p-1)(p-2) + 6p^2 \\ &= p(p+1)(p+2). \end{aligned}$$

Hence the proposition holds recursively at every prime level. □

Now

$$r_p \geq p + 2$$

and

$$s_p \geq p(p+1) + 1.$$

Therefore

$$r_p^3 > p(p+1)(p+2)$$

and

$$r_p s_p > p(p+1)(p+2).$$

Hence

$$\boxed{D_p \geq p(p+1)(p+2).}$$

Combining this with the recursive level bound gives

$$\boxed{\text{closing center} + 1 \leq D_p.}$$

Thus the complete $p$-level lies inside the domain in which the closure information is exhaustive.

### The $p$-center theorem

**Theorem 9.4** ($p$-center theorem)**.** *For every prime $p \geq 5$, the synchronized prime-clock construction produces a completed level containing exactly*

$$\boxed{p}$$

*consecutive certified twin-prime centers.*

*Proof.* The transported zero state generates the two reflected phase orientations.

Removing the single direct closure pair leaves

$$p - 2$$

main positions, and continuation through the two boundaries supplies one terminal completion from each orientation. Hence the completed phase state contains

$$(p-2) + 2 = p$$

outputs.

By the additive–closure correspondence, these are exactly the positions surviving the complete direct and cross-product readouts of the same modular state.

The recursive level bound places every output inside $D_p$. Section 8 therefore guarantees that both neighboring addresses of every retained output are prime.

Finally, the construction proceeds in increasing numerical order, and every intervening six-lattice position not retained is closed by the complete modular information. Hence no twin-prime center between consecutive outputs is omitted.

Thus the completed $p$-level consists of exactly $p$ consecutive certified twin-prime centers. $\square$

## Recursive handoff and infinitude

The final center of one completed level becomes the carried zero of the next:

$$\boxed{\text{closing center at } p = \text{carried zero state of the next level.}}$$

At the same time,

$$pB_p = \prod_{\substack{q \le p \\ q \text{ prime}}} q$$

becomes the complete lower-clock period inherited by the next prime level.

Thus the construction carries forward both parts of the state:

$$\boxed{\begin{gathered} pB_p \longrightarrow \text{next lower-clock period,} \\ \text{closing center} \longrightarrow \text{next additive zero.} \end{gathered}}$$

**Corollary 9.5** (Infinitely many twin primes)**.** *The construction produces infinitely many distinct twin-prime pairs.*

*Proof.* The prime-clock lemma opens a new clock exactly at each prime number. Since there are infinitely many primes, there are infinitely many prime levels.

By the $p$-center theorem, every prime level $p \ge 5$ produces a completed finite list of certified twin-prime centers.

The closing center of each level is strictly greater than its opening center and becomes the opening center of the next. Hence the closing centers form a strictly increasing sequence,

$$42 < 150 < 420 < 858 < 1488 < \cdots .$$

Each closing center is itself a certified twin-prime center. Therefore infinitely many prime levels produce infinitely many distinct twin-prime centers.

For every such center $c$,

$$c - 1 \qquad \text{and} \qquad c + 1$$

are primes differing by 2.

Hence

$$\boxed{\text{there exist infinitely many twin-prime pairs.}}$$

$\square$

The construction is therefore one recursive modular mechanism with two complementary readouts:

$$\boxed{\text{transported additive phase state} \quad\Longleftrightarrow\quad \text{complete closure state.}}$$

The modular zero generates the relative state, the carried additive zero translates it forward, the two reflected orientations determine the completed $p$-member output, and the certification theorem proves that the common retained state consists of genuine consecutive twin-prime centers.

# 10 Executable Realization and Finite Verification

The proof above is independent of computation. Nevertheless, the unified modular construction has a short executable realization that follows the same carried phase arithmetic and provides a finite check of the resulting prime levels.

The program maintains the simultaneous modular state directly.

If the current phase record is

$$R_p(c) = \big(c \bmod q\big)_{q\in P_p},$$

then advancing by an integer displacement $h$ gives

$$R_p(c+h) = R_p(c) + R_p(h),$$

with every component reduced modulo its corresponding prime.

The program therefore does not need to reconstruct the complete phase record from the numerical position at every step. Once a state has been initialized, it can be transported additively.

Its structure is

$$\boxed{\begin{array}{c}\text{recursively generate and advance the prime clocks}\\ \downarrow\\ \text{carry the fixed clock state through the } p^2 \text{ readout}\\ \downarrow\\ \text{form } A_p \text{ and } C_p\\ \downarrow\\ \text{carry the center-phase record on the six-lattice}\\ \downarrow\\ \text{apply direct and cross-product closures}\\ \downarrow\\ \text{locate the } p \text{ outputs of the completed level.}\end{array}}$$

The executable form is:

```
#!/usr/bin/env python3


def add_phase(R, step, P):
    return tuple((r + step) % q for q, r in zip(P, R))


def clocks(n):
```

```
    P = [2]
    R = [0]

    for x in range(3, n + 1):
        R = [(r + 1) % q for q, r in zip(P, R)]

        if all(R):
            P.append(x)
            R.append(0)

    return tuple(P)


def phases(p):
    P = clocks(p)
    x = p
    R = tuple(x % q for q in P)
    A = []

    for _ in range(p * p):
        x += 1
        R = add_phase(R, 1, P)

        if all(R):
            A.append(x)

    return P, tuple(A)


def level(p, start):
    P, A = phases(p)
    C = {a * b for a in A for b in A}

    out = []
    c = start
    R = tuple(c % q for q in P)

    while len(out) < p:
        if (all(r not in (1, q - 1) or c - 1 == q or c + 1 == q
                for q, r in zip(P, R))
                and c - 1 not in C
                and c + 1 not in C):
            out.append(c)

        c += 6
        R = add_phase(R, 6, P)

    return tuple(out)
```

```
def run(n):
    start = 2 * 3

    for p in clocks(n)[2:]:
        L = level(p, start)
        print(p, L)
        start = L[-1]


if __name__ == "__main__":
    import sys
    run(int(sys.argv[1]) if len(sys.argv) > 1 else int(input("Prime ceiling: ")))
```

### Recursive prime clocks

The function `clocks` is a direct executable form of the recursive clock rule of Section 3.

It begins with the mod-2 clock at its zero phase:

```
P = [2]
R = [0]
```

At each successive integer, every already opened clock advances by one phase:

```
R = [(r + 1) % q for q, r in zip(P, R)]
```

A new clock is opened exactly when every previous clock has nonzero phase:

```
if all(R):
P.append(x)
R.append(0)
```

The newly opened clock is appended at phase 0, exactly as in the mathematical construction.

By the prime-clock lemma, this occurs precisely at the prime positions.

Thus the program does not begin from a supplied prime table. The sequence

$$2,\ 3,\ 5,\ 7,\ 11,\ 13, \ldots$$

is generated recursively by the moving clock state itself.

### Additive phase transport

The helper function

```
def add_phase(R, step, P):
return tuple((r + step) % q for q, r in zip(P, R))
```

implements the phase-addition identity

$$\boxed{R_p(c+h) = R_p(c) + R_p(h).}$$

It is used in both finite readouts of the program.

In `phases`, the fixed clock state advances by ordinary integer steps,

$$h = 1.$$

In `level`, the center state advances on

$$\mathcal{L}_6$$

by

$$h = 6.$$

The value 6 here is only the spacing of the center lattice. It is not being introduced as one of the inherited open phase addresses.

The carried phase information described in the proof—beginning with

$$0,\ 12,\ 18$$

and continuing through completed quantities such as

$$30,\ 42,\ 66,\ 78, \ldots$$

is the compressed additive structure produced by repeated transport of this same simultaneous phase record.

Thus the expanded clock motion and the compressed additive map are two resolutions of the same arithmetic.

## The finite address readout

The function `phases` begins at the prime level $p$ with the complete phase record

$$R_p(p).$$

It then carries that state through exactly

$$p^2$$

ordinary integer ticks.

Whenever every component is nonzero,

```
if all(R):
A.append(x)
```

the address is recorded.

By the finite open-address theorem, the resulting set is exactly

$$\boxed{A_p = \{r : r \text{ prime},\ p < r \le p(p+1)\}.}$$

The function `level` then forms

$$C_p = \{ab : a, b \in A_p\}.$$

Thus the factor information used for higher-order closure is generated from the same advancing clock state.

## The completed center level

The function `level` begins from the closing center carried from the preceding level:

```
c = start
R = tuple(c % q for q in P)
```

This initializes the numerical representative of the carried state.

The program then advances on the six-lattice:

```
c += 6
R = add_phase(R, 6, P)
```

This should not be interpreted as the mathematical proof merely walking through multiples of 6 and asking whether their neighbors are prime.

No primality test is performed on those neighbors.

Instead, the numerical loop is an executable readout of the already constructed modular state. The phase record is transported additively, and the two closure mechanisms derived in the proof are applied to that record.

The first condition

```
all(r not in (1, q - 1)
or c - 1 == q or c + 1 == q
for q, r in zip(P, R))
```

implements direct closure.

For an active prime $q$,

$$r = 1$$

places $c - 1$ on the zero phase, while

$$r = q - 1$$

places $c + 1$ on the zero phase.

The two explicit exceptions occur when that neighboring address is the prime $q$ itself rather than a composite multiple of $q$.

The remaining conditions

```
c - 1 not in C
and c + 1 not in C
```

implement the higher-order cross-product closures.

Hence the computational acceptance condition is exactly

$$\boxed{\text{survive the direct clock state} \quad + \quad \text{survive the cross-product state.}}$$

By the additive–closure correspondence established in Section 9, this closure-side readout identifies the same retained positions represented by the completed additive phase map.

### Why the loop terminates at $p$

The line

```
while len(out) < p:
```

does not assume the existence of $p$ twin-prime centers.

The preceding $p$-center theorem has already proved that the completed relative phase state contains exactly

$$p$$

retained outputs and that all of them occur before the certification boundary $D_p$.

The loop therefore has only a computational stopping role: it terminates once the numerical representatives of that completed $p$-member state have been located.

Similarly, the program does not calculate $D_p$.

The certification boundary is required by the proof, not by the execution. Its role is to establish that the complete $p$-level lies inside a domain in which the direct and cross-product closure information is exhaustive.

### Finite output

The first levels returned by the recurrence are

| $p$ | retained centers |
|---|---|
| 5 | 6, 12, 18, 30, 42 |
| 7 | 42, 60, 72, 102, 108, 138, 150 |
| 11 | 150, 180, 192, 198, 228, 240, 270, 282, 312, 348, 420 |
| 13 | 420, 432, 462, 522, 570, 600, 618, 642, 660, 810, 822, 828, 858. |

The next level begins from the preceding closing center:

$$\begin{aligned} p = 17: \qquad & 858, 882, 1020, 1032, 1050, 1062, 1092, 1152, \\ & \quad 1230, 1278, 1290, 1302, 1320, 1428, 1452, 1482, 1488. \end{aligned}$$

Several successive closing centers are

| $p$ | closing center |
|---|---|
| 5 | 42 |
| 7 | 150 |
| 11 | 420 |
| 13 | 858 |
| 17 | 1488 |
| 19 | 2238 |
| 23 | 3372 |
| 29 | 4518 |
| 31 | 6300. |

Each closing center becomes the carried opening state of the next level.

Thus

$$42 \longrightarrow 150 \longrightarrow 420 \longrightarrow 858 \longrightarrow 1488 \longrightarrow \cdots$$

is not supplied to the program separately. It is generated recursively from the final retained output of each preceding level.

The finite outputs agree with independently generated consecutive twin-prime centers through the levels examined.

That comparison is a verification of the executable realization, not a premise of the proof.

The logical direction remains

$$\boxed{\text{proved modular construction} \longrightarrow \text{completed retained phase state} \longrightarrow \text{program locates its numerical representativ}}$$

rather than

$$\text{program finds examples} \longrightarrow \text{existence proof.}$$

The program therefore provides a compact computational realization of the same additive modular and closure machinery used in the proof.

**Code availability.** The Python implementation used in this section is available at `https://github.com/ProsperousPlanet/primecenters/blob/main/transport12.py`.

## 11 Discussion and Conclusion

The construction developed in this paper treats the Twin Prime Conjecture as a recursive problem in synchronized modular phase information.

The first two prime clocks reduce every nonexceptional twin-prime center to the six-spaced lattice

$$\mathcal{L}_6 = \{6, 12, 18, 24, \ldots\}.$$

From that point onward, no external list of primes is supplied to the construction. New prime clocks are opened recursively by the same zero-phase rule, and the accumulated clock state is carried from one prime level into the next.

The central feature of the construction is that this state admits two complementary descriptions.

The first is the *additive phase map*. The surviving numerical phase information can be carried and added forward. The initial completed state leaves

$$0,\ 12,\ 18$$

as its open phase addresses, and later completed quantities such as

$$30,\ 42,\ 66,\ 78,\ldots$$

arise from the prime clocks already present in the system.

In this representation, the additive zero is not an empty state. It is the carried completed state from which the two forward phase orientations are measured. The closing center of one level therefore becomes the carried zero of the next.

The second description is the *closure map*. Direct zero phases of the active prime clocks remove positions whose neighboring addresses contain an already opened prime factor. The finite clock readout also generates

$$A_p,$$

the prime addresses required at the current level, and their pairwise products

$$C_p$$

supply the remaining higher-order composite closures.

These two descriptions are not independent mechanisms.

The additive map determines how the surviving phase state is assembled and transported, while the closure map determines which numerical placements of that state are impossible because one of the neighboring addresses is composite.

Thus the same synchronized clocks are being read in two directions:

$$\boxed{\begin{array}{c}\text{additive phase structure}\\ \text{and}\\ \text{composite closure structure}\end{array}}$$

are two representations of one recursive modular state.

The 11-level displays this correspondence explicitly. The direct clocks first leave seventeen possible centers between the carried opening 150 and the closing center 420. The cross-product state closes six of them, leaving

$$150,\ 180,\ 192,\ 198,\ 228,\ 240,\ 270,\ 282,\ 312,\ 348,\ 420.$$

The additive phase construction produces exactly the same eleven positions.

This equality is more significant than a numerical coincidence. It identifies the forward additive construction with the survivor set of the complete closure construction.

At a general prime level $p$, the new mod-$p$ clock contains exactly $p$ relative phases. The two phases adjacent to zero,

$$1 \qquad \text{and} \qquad p-1,$$

are the direct divisibility boundaries.

The remaining reflected phase structure contains

$$p - 2$$

main positions. Because the inherited lower-clock state persists when either boundary is reached, the two orientations each retain one terminal completion.

The completed state therefore contains

$$\boxed{(p - 2) + 2 = p}$$

retained outputs.

This is not a statistical estimate of how many twin primes should remain. It is the finite population of the completed relative phase state itself.

The certification boundary

$$D_p$$

plays a different role. It neither creates the additive state nor determines its closing center. Rather, it proves that the direct clocks and the finite cross-product information are exhaustive throughout a domain containing the complete $p$-level.

The recursive level bound establishes

$$\text{closing center} + 1 \leq D_p.$$

Consequently every retained output of the completed additive state lies inside the region where survival under the two closure mechanisms is equivalent to having two prime neighbors.

The $p$-center theorem therefore identifies the completed output

$$c_1 < c_2 < \cdots < c_p$$

as $p$ consecutive twin-prime centers.

The distinction between the various completed numerical states is also important. A quantity such as

$$pB_p$$

marks the full simultaneous completion of the prime clocks through $p$, whereas the closing center records the terminal retained output of the corresponding additive level. The certification boundary $D_p$ is larger again and records the finite range through which the available factor information is exhaustive.

These quantities serve different purposes and need not coincide.

What passes recursively from one level to the next is therefore not a single number alone. Two related parts of the state are inherited:

$$\boxed{\begin{aligned} pB_p &\longrightarrow \text{complete lower-clock period of the next prime level,} \\ \text{closing center} &\longrightarrow \text{carried additive zero of the next prime level.} \end{aligned}}$$

This produces the observed sequence of closing centers

$$42 \longrightarrow 150 \longrightarrow 420 \longrightarrow 858 \longrightarrow 1488 \longrightarrow \cdots,$$

without supplying that sequence independently.

The executable recurrence included in the paper provides a compact finite realization of the same construction. It recursively generates the prime clocks, carries their phase records forward, forms the finite address and cross-product information, applies the closure readout, and locates the numerical representatives of the completed $p$-member state. Its agreement with independently generated twin-prime data at the examined levels provides a computational check, but the program is not used as a premise of the proof.

The Python implementation is available at
https://github.com/ProsperousPlanet/primecenters/blob/main/transport12.py.

The broader structural point is that the construction does not rely on a density estimate, a probabilistic expectation, or the assumption that sufficiently many twin primes happen to survive.

Instead, the prime clocks generate the phase structure, the additive representation organizes that structure into a forward sequence, and the closure representation removes exactly the composite neighboring cases. The two descriptions meet in the same retained center set.

Finally, the recursive prime-clock rule opens a new clock at every prime, and there are infinitely many primes. Hence there are infinitely many prime levels.

By the $p$-center theorem, every prime level $p \geq 5$ completes with $p$ certified twin-prime centers. Its closing center is strictly greater than its opening center and becomes the carried opening of the next level.

The resulting closing centers therefore form a strictly increasing sequence of certified twin-prime centers.

Each such center $c$ determines the prime pair

$$(c-1, c+1),$$

whose members differ by 2.

Therefore the recursive construction yields infinitely many distinct twin-prime pairs:

$$\boxed{\text{There exist infinitely many twin primes.}}$$

The proof may thus be summarized by the single recursive correspondence

$$\boxed{\begin{array}{c} \text{prime clocks} \\ \downarrow \\ \text{carried modular phase state} \\ \downarrow \\ \text{additive phase representation} \iff \text{direct and cross-product closure representation} \\ \downarrow \\ p \text{ consecutive certified twin-prime centers} \\ \downarrow \\ \text{closing center carried to the next prime level} \\ \downarrow \\ \text{infinitely many distinct twin-prime pairs.} \end{array}}$$

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